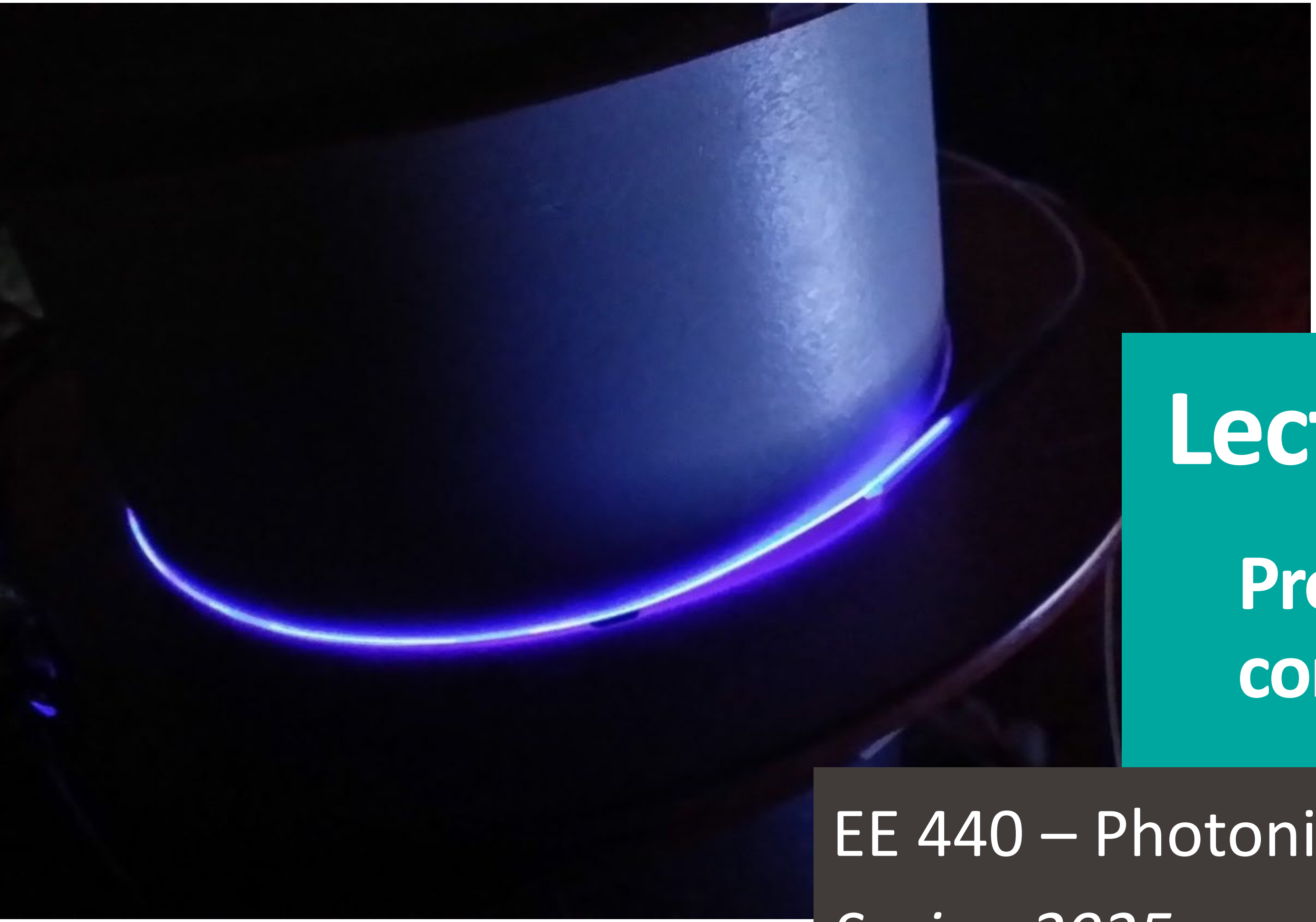

Lecture 6

Propagation and dispersion compensation

EE 440 – Photonic systems and technology

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Lecture 4 outline

Basic propagation equation

- Chromatic dispersion
- Loss
- Nonlinearity
- $\Rightarrow$ Nonlinear Schrödinger equation

Propagation of chirped pulses

- Effect on the spectrum
- Compression

Higher order dispersion

Fourier transform definition

The textbook uses the following definition:

$$\tilde{s}(\omega) = \int_{-\infty}^{\infty} s(t)\exp(i\omega t)dt \qquad s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{s}(\omega)\exp(-i\omega t)d\omega$$

- Note: one of many different definitions used in different field of science

From this definition, it follows that:

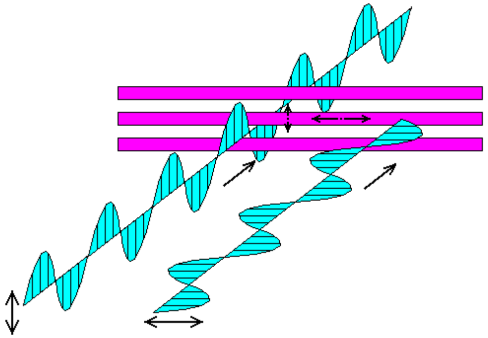
$$\frac{\partial}{\partial t} \Leftrightarrow -i\omega \qquad \exp(i\Omega t)\, s(t) \Leftrightarrow \tilde{s}(\omega + \Omega)$$

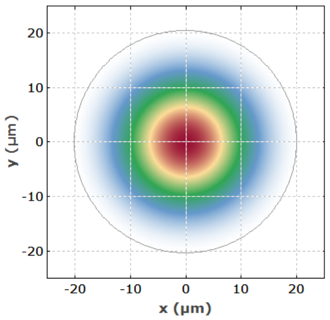
Propagation equation

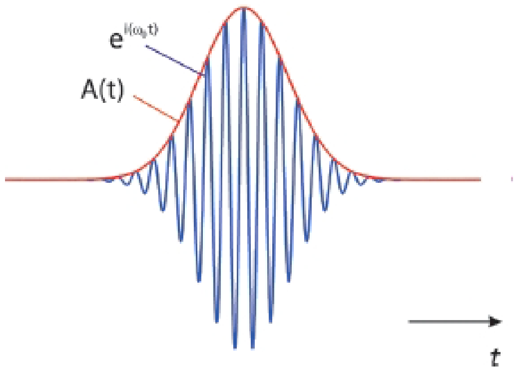
Let’s start from the beginning

A wave traveling in the positive z-direction in a single mode fiber is described by:

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}[\underbrace{\hat{\boldsymbol{e}}}_{\text{Polarization unit vector}}\underbrace{F(x,y)}_{\text{Spatial profile of the mode (transverse)}}\underbrace{A(z,t)}_{\text{Slowly varying amplitude of the pulse envelope}}\exp(i\underbrace{\beta_0}_{\text{Mode propagation constant at }\omega_0}z - i\omega_0 t)]$$







Fourier domain

Only $A(z,t)$ changes upon propagation.

- An optical signal can always be considered the sum of monochromatic waves, by Fourier decomposition.

$$A(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{A}(z,\Delta\omega)\exp(-i\Delta\omega t)d\Delta\omega$$

$$\Delta\omega = \omega - \omega_0$$

- Each frequency component ω in the material propagates with a slightly different propagation constant, $\mathrm{n}(\omega)$.

Fourier domain

Let’s consider one frequency component at ω : $\tilde{A}(z, \Delta\omega)$

- This frequency component will propagate inside the fiber with a propagation constant $\beta_p(\omega)$ that is different from the propagation constant β_0 of the central frequency ω_0.

$$\boxed{\tilde{A}(z, \Delta\omega) = \underbrace{\tilde{A}(0, \Delta\omega)}_{\substack{\text{Fourier transform of} \\ \text{the initial envelope} \\ \mathrm{A}(0,t)}} \underbrace{\exp\left[i\beta_p(\omega)z - i\beta_0 z\right]}_{\substack{\text{Phase difference accumulated} \\ \text{after propagation } z \text{ between } \omega \\ \text{and } \omega_0}}}$$

Propagation constant

Up to now we have seen that the propagation constant $\beta_p(\omega)$ depends on the free space wavenumber k_0 and the effective refractive index.

- This is for a purely real propagation constant

In general, $\beta_p(\omega)$ is complex: $\beta_p(\omega) = \beta_{p,real}(\omega) + i\beta_{p,im}(\omega)$

- The real part influences the phase of the light as it propagates
- The imaginary part influences the amplitude of the light as it propagates

$$\exp[i\beta_p z] \Rightarrow \exp[i(\beta_{p,real} + i\beta_{p,im})z]$$

$$\exp[i\beta_p z] \Rightarrow \exp[i\beta_{p,real} z]\exp[-\beta_{p,im} z]$$

Phase evolution

Amplitude evolution
For $\beta_{p,im} > 0$ loss

Effective refractive index

In addition, the real part of the propagation constant depends on the effective refractive index

$$\beta_{p,real}(\omega) = \bar{n}(\omega)\frac{\omega}{c_0}$$

The index in a material can have some fluctuations around the linear contribution due to nonlinearities. We can therefore express it as:

$$\bar{n}(\omega) = [\bar{n}_{lin}(\omega) + \delta n_{NL}(\omega)]$$

Propagation constant

$$\beta_p(\omega) = \beta_{p,real} + i\beta_{p,im}$$

$$\beta_p(\omega) = [\bar{n}_{lin}(\omega) + \delta n_{NL}(\omega)]\frac{\omega}{c_0} + i\frac{\alpha(\omega)}{2}$$

$$\beta_p(\omega) \approx \underbrace{\beta_L(\omega)}_{\text{Linear part}} + \underbrace{\beta_{NL}(\omega_0)}_{\text{Nonlinear part}} + \underbrace{i\frac{\alpha(\omega_0)}{2}}_{\text{Fiber loss}}$$

Linear part
$$\beta_L(\omega) = \bar{n}(\omega)k_0$$

Nonlinear part
$$\beta_{NL}(\omega) = \gamma(\omega)|A|^2$$

Fiber loss

$$\gamma(\omega) = \frac{2\pi n_2}{\lambda_0 A_{\text{eff}}}$$

n_2 : Kerr index (material property) in m^2/W

A_{eff} : Effective area (Waveguide property)

Linear part of propagation constant, $\beta_L(\omega)$

Can expand $\beta_L(\omega)$ in a Taylor series around ω_0

$$\beta_L(\omega) \approx \beta_0 + \beta_1 \Delta\omega + \frac{\beta_2}{2}\Delta\omega^2 + \frac{\beta_3}{6}\Delta\omega^3 \ldots$$

$$\beta_m = \left(\frac{d^m\beta}{d\omega^m}\right)_{\omega=\omega_0}$$

Recall: β_0 is related inversely to the phase velocity v_p

β_1 is related inversely to the group velocity v_g of the pulse

β_2 is related to the dispersion parameter $D = \frac{d}{d\lambda}\left(\frac{1}{v_g}\right)$

β_3 is related to the dispersion slope $S = \frac{dD}{d\lambda}$

Basic propagation equation

We are looking for the equation that governs the evolution of $A(z,t)$

$$\Rightarrow A(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{A}(z,\Delta\omega)\exp(-i\Delta\omega t)d\Delta\omega$$

Spectral decomposition

$$\Rightarrow \tilde{A}(z,\Delta\omega) = \tilde{A}(0,\Delta\omega)\exp\left[i\beta_p(\omega)z - i\beta_0 z\right]$$

Phase shift between ω, ω_0

$$\Rightarrow \beta_p(\omega) = \beta_L(\omega) + \gamma(\omega)|A|^2 + i\frac{\alpha(\omega_0)}{2}$$

Propagation constant

$$\Rightarrow \beta_p(\omega) \approx \beta_0 + \beta_1\Delta\omega + \frac{\beta_2}{2}\Delta\omega^2 + \frac{\beta_3}{6}\Delta\omega^3 \ldots$$

Linear part

Take partial derivative with respect to z: $\frac{\partial A}{\partial z}$

Write in the time domain by using $\Delta\omega \Leftrightarrow i\frac{\partial}{\partial t}$

Basic propagation equation

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial t^3} = i\gamma|A|^2 A - \frac{\alpha}{2}A$$

Introduce the new variables $t' = t - \beta_1 z$ (retarded time) and $z' = z$

- i.e. coordinate system that moves with the pulse group velocity
- Primes are implicit

$$\frac{\partial A}{\partial z} + i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} - \frac{\beta_3}{6}\frac{\partial^3 A}{\partial t^3} = i\gamma|A|^2 A - \frac{\alpha}{2}A$$

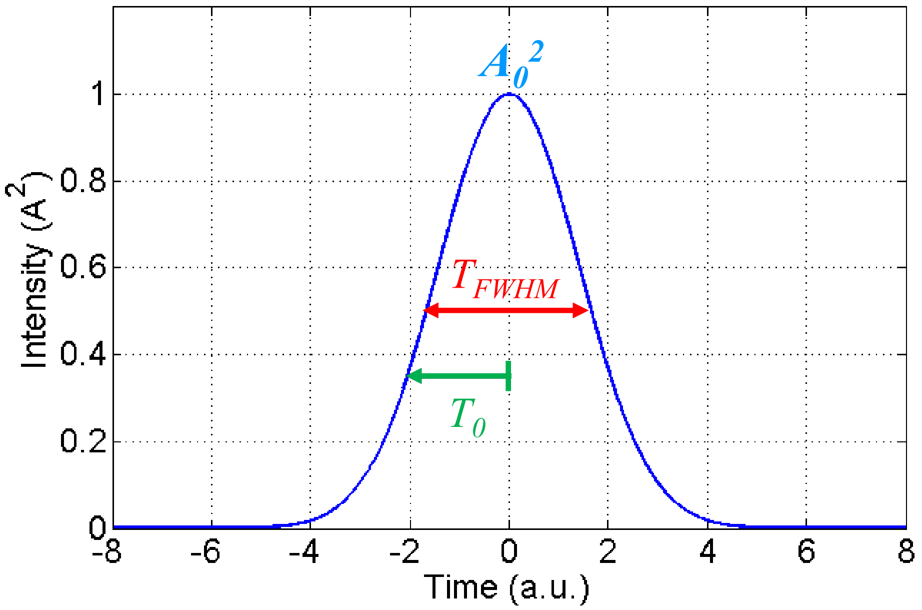
Nonlinear Schrödinger equation (NLSE)

Propagation of chirped Gaussian pulse

Propagation of a chirped Gaussian pulse

Initial field is:

$$A(0,t) = A_0 \exp\left[-\frac{1+iC}{2}\left(\frac{t}{T_0}\right)^2\right]$$



A_0 : the peak amplitude

T_0 : the *half width at the 1/e* intensity point. It is related to T_{FWHM}

$$T_{FWHM} = \sqrt{2\ln 2}\,T_0 \approx 1.665T_0$$

C: governs the frequency chirp imposed on the pulse (chirp parameter)

Recall on Chirp

A pulse is chirped if the carrier frequency changes with time.

Frequency change (or chirp frequency) $\delta\omega_c(t)$ is related to the derivative of the phase:

$$\boxed{\delta\omega_c(t) = -\frac{\partial\phi}{\partial t}}$$

For our Gaussian pulse we have:

$$A(0,t) = A_0 \exp\left[-\frac{1+iC}{2}\left(\frac{t}{T_0}\right)^2\right] = A_0 \exp\left[-\frac{1}{2}\left(\frac{t}{T_0}\right)^2\right] \exp\left[-i\frac{C}{2}\left(\frac{t}{T_0}\right)^2\right]$$

$$\delta\omega_c(t) = -\frac{\partial}{\partial t}\left[-\frac{C}{2}\left(\frac{t}{T_0}\right)^2\right] = \frac{C}{T_0^2}t$$

$C < 0$: down chirp

$C > 0$: up chirp

Spectrum of chirped Gaussian pulses

Let $\tilde{A}(0, \Delta\omega)$ be the Fourier transform of $\mathrm{A}(0, t)$.

It can be found using the following expression: $\int_{-\infty}^{\infty} \exp(-ax^2 - 2bx) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{a}\right)$

We obtain: $\tilde{A}(0, \Delta\omega) = A_0 \left(\frac{2\pi T_0^2}{1 + iC}\right)^{1/2} \exp\left[-\frac{\omega^2 T_0^2}{1(1 + iC)}\right]$

The 1/e spectral half width (intensity) is $\Delta\omega_0 = \sqrt{1 + C^2} \big/ T_0$

The time-bandwidth product (at 1/e point) is $\Delta\omega_0 T_0 = \sqrt{1 + C^2}$

When $C = 0$, the pulses are chirp free and said to be **transform-limited**:
Chirping broadens the pulse bandwidth for a given pulse width.

Propagation of chirped Gaussian pulses

To study dispersion, let's first:

- neglect nonlinearities and losses ($\gamma = 0, \alpha = 0$)
- assume that the wavelength is far from the ZDW ($\beta_3 \approx 0$, β_2 dominates)

$$\frac{\partial A}{\partial z} + i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = 0$$

Use the Fourier transform method, general solution takes the form:

$$\tilde{A}(z, \Delta\omega) = \tilde{A}(0, \Delta\omega)\exp\left[i\frac{\beta_2}{2}\Delta\omega^2 z\right]$$

And $A(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{A}(0,\Delta\omega)\exp\left(i\frac{\beta_2}{2}\Delta\omega^2 z - i\Delta\omega t\right)d\Delta\omega$

Propagation of chirped Gaussian pulse

Using the derived expression for $\tilde{A}(0, \Delta\omega)$ we can use the same identity trick to get $A(z, t)$. We can show that:

$$A(z,t) = \frac{A_0}{\sqrt{Q(z)}} \exp\left[-\frac{(1+iC)t^2}{2T_0^2 Q(z)}\right] \quad \text{with} \quad Q(z) = 1 + (C - i)\frac{\beta_2 z}{T_0^2}$$

Gaussian pulse remains Gaussian on propagation
Its width, chirp and amplitude evolve as the pulse propagates

Changes in pulse width (Gaussian pulse)

Changes in pulse width with z are quantified through broadening factor:

- Pulse width after propagation compared to input pulse width.

$$\frac{T(z)}{T_0} = \left[\left(1 + \frac{C\beta_2 z}{T_0^2}\right)^2 + \left(\frac{\beta_2 z}{T_0^2}\right)^2\right]^{1/2}$$

We also define the dispersion length L_D as the distance at which an unchirped pulse broadens by factor of $\sqrt{2}$:

$$L_D = \frac{T_0^2}{|\beta_2|}$$

Changes in pulse width (Gaussian pulse)

- Unchirped pulse: $C = 0$

$$\frac{T(z)}{T_0} = \left[1 + \left(\frac{\beta_2 z}{T_0^2}\right)^2\right]^{1/2} = \left[1 + \left(\frac{z}{L_D}\right)^2\right]^{1/2} > 1 \quad \text{For all } z$$

Pulse broadens

- Chirped pulse case 1: $C\beta_2 > 0$

$$\frac{T(z)}{T_0} = \left[\left(1 + \frac{C\beta_2 z}{T_0^2}\right)^2 + \left(\frac{\beta_2 z}{T_0^2}\right)^2\right]^{1/2} > 1 \quad \text{For all } z$$

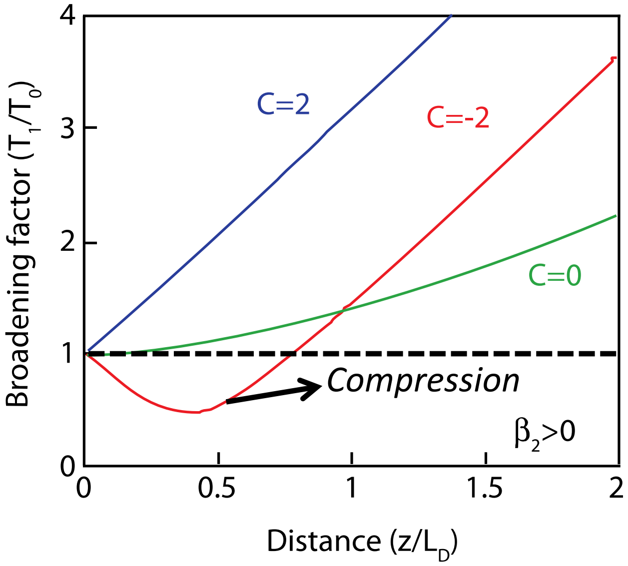
Pulse broadens faster than for $C = 0$

Changes in pulse width

- Chirped pulse case 2: $C\beta_2 < 0$

$$\frac{T(z)}{T_0} = \left[\left(1 + \frac{C\beta_2 z}{T_0^2}\right)^2 + \left(\frac{\beta_2 z}{T_0^2}\right)^2\right]^{1/2} \Rightarrow \frac{T(z)}{T_0} = \left[\left(1 - \frac{|C|z}{L_D}\right)^2 + \left(\frac{z}{L_D}\right)^2\right]^{1/2}$$

Pulse will initially compress!



Pulse compression

In the case $C\beta_2 < 0$ the pulse width reaches minimum after a propagation distance given by:

$$z_{min} = \frac{|C|}{(1+C^2)} L_D$$

The minimum value is

$$T_{min} = \frac{T_0}{\sqrt{1+C^2}} = \frac{1}{\Delta\omega_0}$$

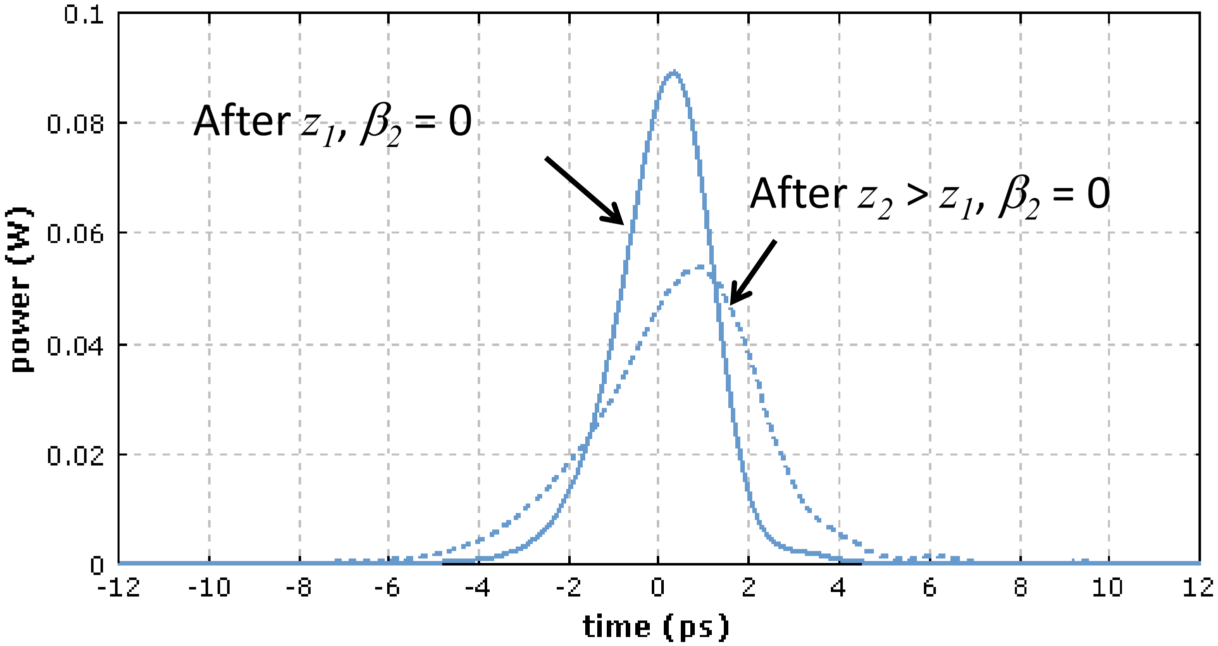
Beyond that point, chirped pulse broadens faster than unchirped one.

Non-Gaussian pulses

Only Gaussian pulses remain Gaussian upon propagation

In addition, if β_3 (higher order dispersion) cannot be ignored even Gaussian pulses will not remain Gaussian ….

- Observe a tail with oscillatory behavior



Non-Gaussian pulses

In these cases, the FHWM is not a good measure of pulse width

- The RMS (root mean square) pulse width σ can be used instead

$$\sigma = \sqrt{\langle t^2 \rangle} \qquad \text{with} \qquad \sqrt{\langle t^m \rangle} = \frac{\int_{-\infty}^{\infty} t^m |A(z,t)|^2 dt}{\int_{-\infty}^{\infty} |A(z,t)|^2 dt}$$

- σ can be calculated when the initial pulse is known

For *input Gaussian pulse*, its initial RMS pulse width σ_0 is related to the 1/e intensity point (T_0) as

$$\boxed{\sigma_0 = \frac{T_0}{\sqrt{2}}}$$

Effect of source spectrum width

Up to now, considered sources with a narrow spectral width

- i.e. much less than the pulse spectral width

This is not always the case !

- For example, if LEDs are used as light sources
- Will lead to strong dispersive broadening…

For Gaussian source spectrum (RMS spectral width σ_ω) and Gaussian pulses we define $\sigma_\omega = 2\sigma_\omega\sigma_0$

- Source with large spectral width : V_ω >> 1
- Source with small spectral width: V_ω << 1

$$\frac{\sigma^2(z)}{\sigma_0^2} = \left(1 + \frac{C\beta_2 z}{2\sigma_0^2}\right)^2 + (1 + V_\omega^2)\left(\frac{\beta_2 z}{2\sigma_0^2}\right)^2 + (1 + C^2 + V_\omega^2)^2\left(\frac{\beta_3 z}{4\sqrt{2}\sigma_0^3}\right)^2$$

General rule on limitation on bit rate

In the limit of large broadening and using the common criteria for the bit (baud if non binary) rate, the maximum allowed width after propagation is:

$$4\sigma \leq T_B$$

$$\sigma \leq \frac{1}{4B}$$

Example: incoherent source

Let’s consider an LED, unchirped and operating far from the ZDW

Let’s show that the expression for the bandwidth-distance limit in the limit of large broadening is given by (σ_λ the RMS spectral width in wavelength)

$$4BL|D|\sigma_\lambda \leq 1$$

Dispersion compensation

Dispersion recall

The total dispersion D is the sum of the waveguide and material contributions

- Dispersion can be negative, positive and equal to zero at the wavelength typically denoted λ_{ZDW}.

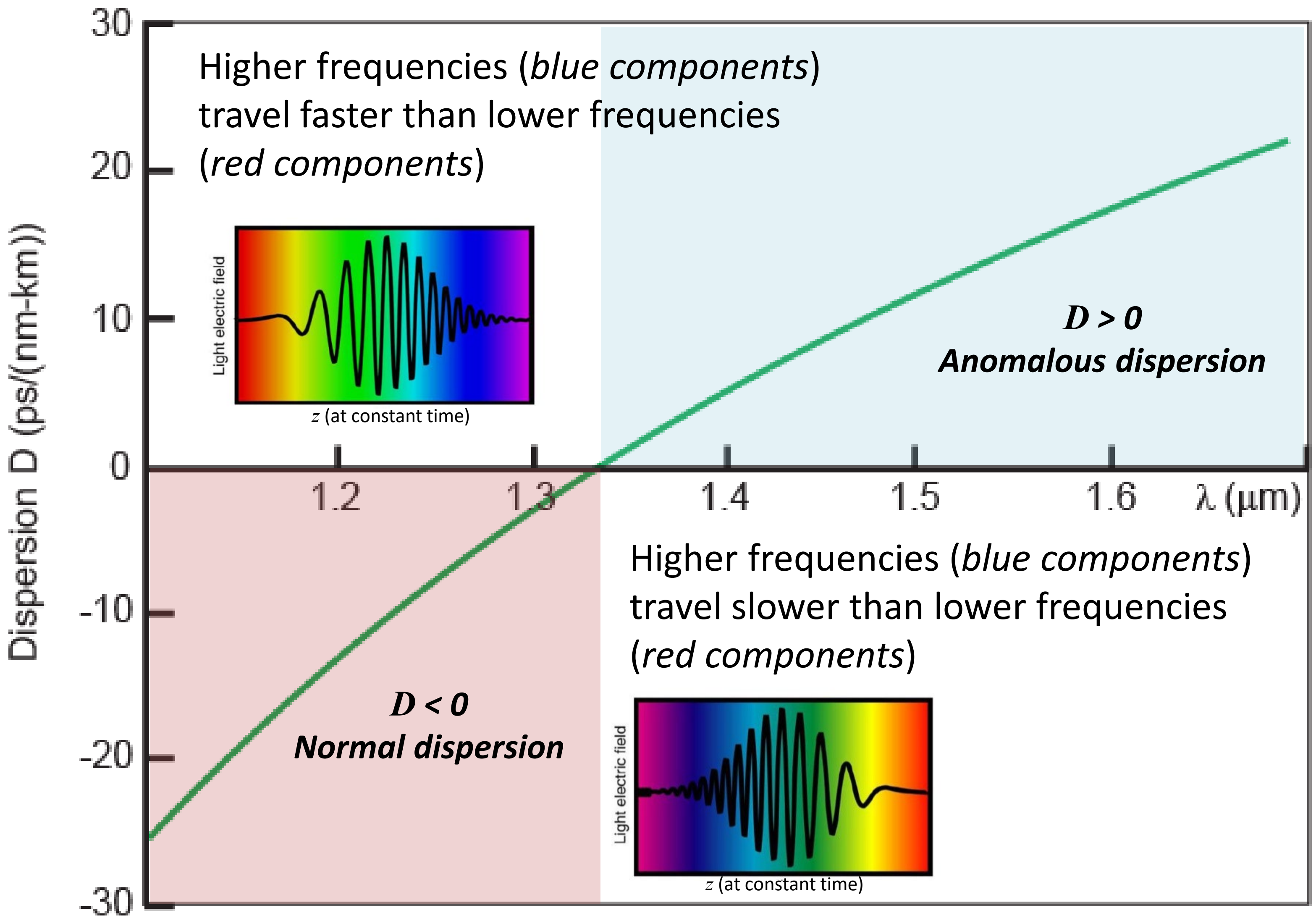
In standard single mode fiber (SMF)

$D < 0$ for $\lambda < 1.31$ μm: "**normal dispersion**", the group velocity of higher frequencies is lower than for lower frequencies

$D > 0$ for $\lambda > 1.31$ μm: "**anomalous dispersion**", the group velocity of higher frequencies is higher than for lower frequencies

- Note: pulses are affected differently by nonlinear effects in these two cases

Total dispersion



Higher frequencies (*blue components*) travel faster than lower frequencies (*red components*)

***D* > 0**
Anomalous dispersion

Higher frequencies (*blue components*) travel slower than lower frequencies (*red components*)

***D* < 0**
Normal dispersion

Dispersion problem and solutions

Using optical amplification, dispersion (not loss) is the major limitation.

- In general, dispersion is important at bit rates > 5 Gbit/s.
- Even if the source is chirp-free and the fiber is single-mode.

Dispersion must be compensated for.

- Then noise and nonlinearities become the major limitations.

Compensation can be in:

- Optical domain: DCF, FBG, filters, OPC, and solitons.
- Electrical domain: Pre- or post-compensation, often using DSP.

Aim of dispersion compensation is to cancel the phase factor:

$$A(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(0,\omega)\exp\left[\frac{i}{2}\beta_2 z\omega^2 + \frac{i}{6}\beta_3 z\omega^3 - i\omega t\right]d\omega$$

Compensation in the optical domain

In general, an optical device with field transfer function:

$$H(\omega) = |H(\omega)| \exp[i\phi(\omega)] = |H(\omega)| \ \exp\left[i\left(\phi_0 + \phi_1\omega + \frac{1}{2}\phi_2\omega^2 + \frac{1}{6}\phi_3\omega^3\right)\right]$$

Will modify the electric field such that:

$$A(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(0,\omega)H(\omega)\exp\left[\frac{i}{2}\beta_2 z\omega^2 + \frac{i}{6}\beta_3 z\omega^3 - i\omega t\right] d\omega$$

The dispersion is perfectly canceled if:

$$\phi_2 = -\beta_2 L \qquad \phi_3 = -\beta_3 L \qquad |H(\omega)| = 1$$

- ϕ_0 only changes the absolute phase: is of no consequence
- ϕ_1 introduces a delay: important to keep small to avoid latency

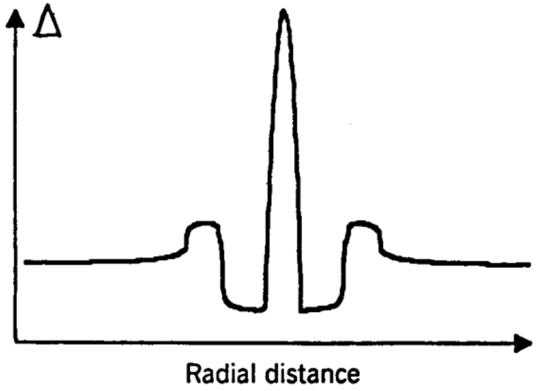
Dispersion compensating fiber (DCF)

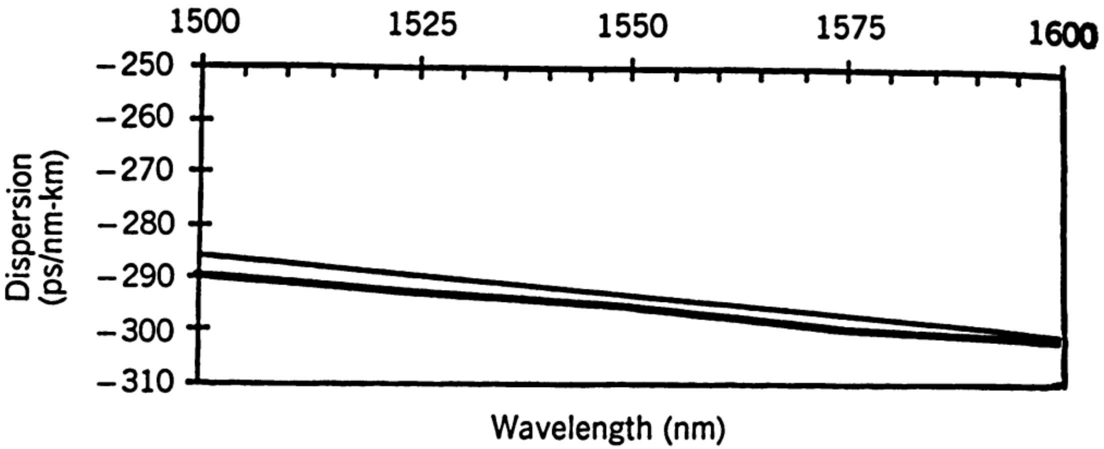
For most telecom wavelengths (>1.3μm) , SMF has a positive dispersion

- Can be compensated for by inserting a fiber with negative dispersion (i.e. with large negative D_{WG})

DCF can be made to have a very strong normal dispersion:

- D of -100 to -300 ps/(nm.km)
- Loss is relatively high (0.4 to 1 dB/km)
- The core is small, the nonlinear coefficient is relatively large





Dispersion maps

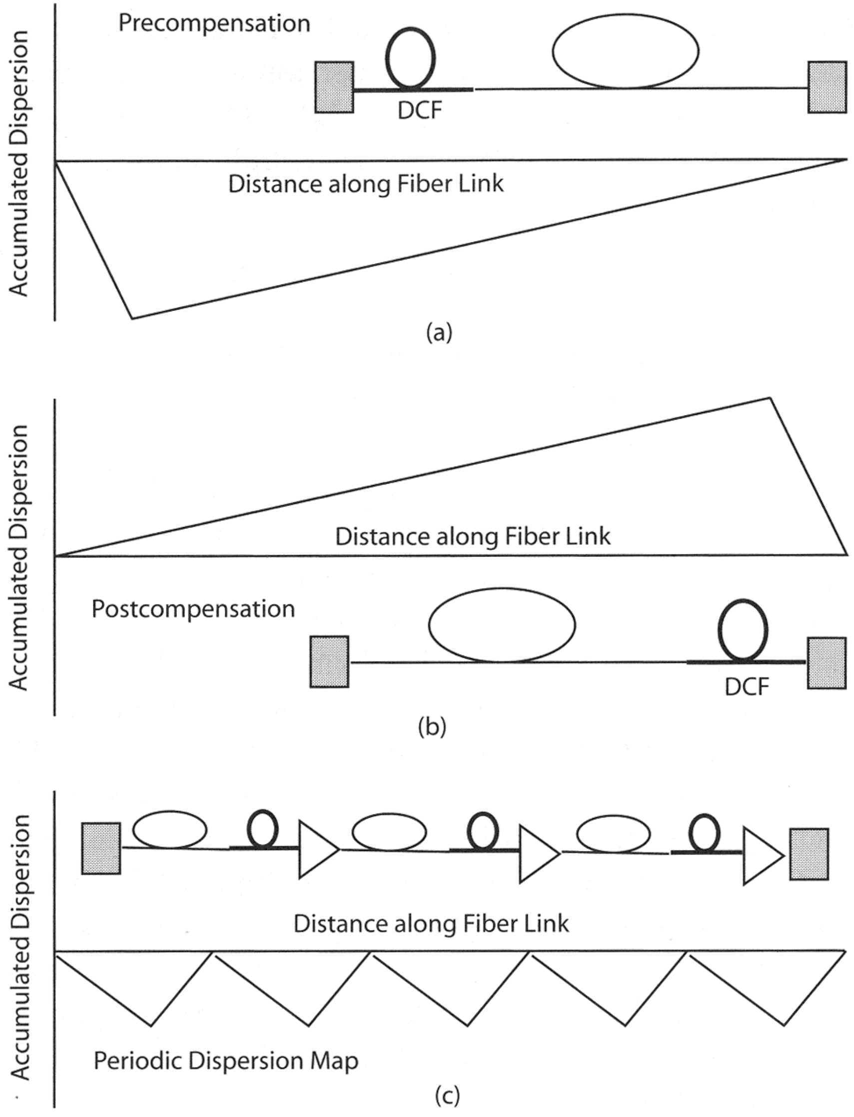
DCFs can be placed in different ways

Can have three dispersion maps

- Pre-compensation
- Post-compensation
- Periodic compensation

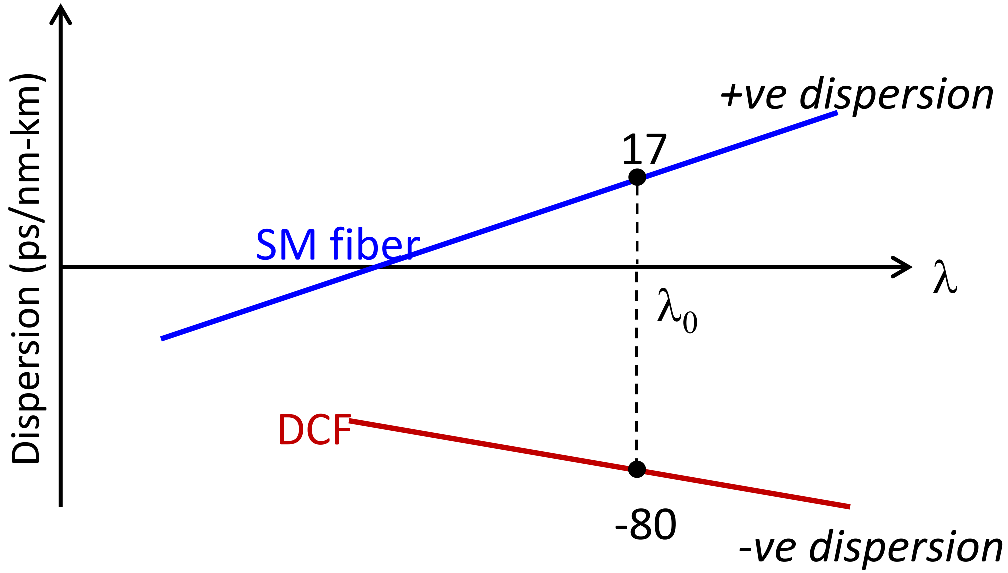
In practice, periodic compensation is often used

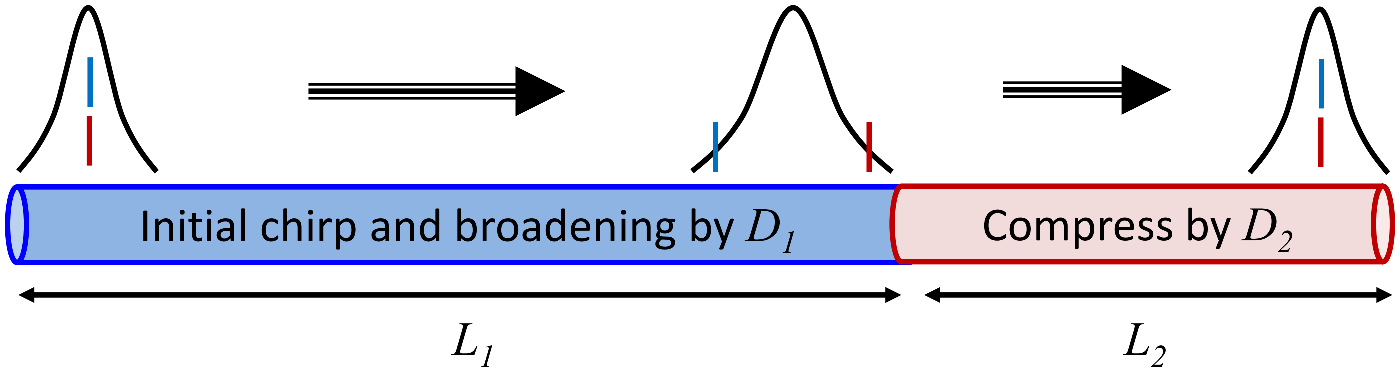
Including nonlinear effects, performance can vary significantly



Fixed compensation: DCF

Concept: use a span of fiber to *compress* an initially chirped pulse





Slope compensation

Conditions for perfect dispersion compensation

$$D_2 L_2 + D_1 L_1 = 0$$

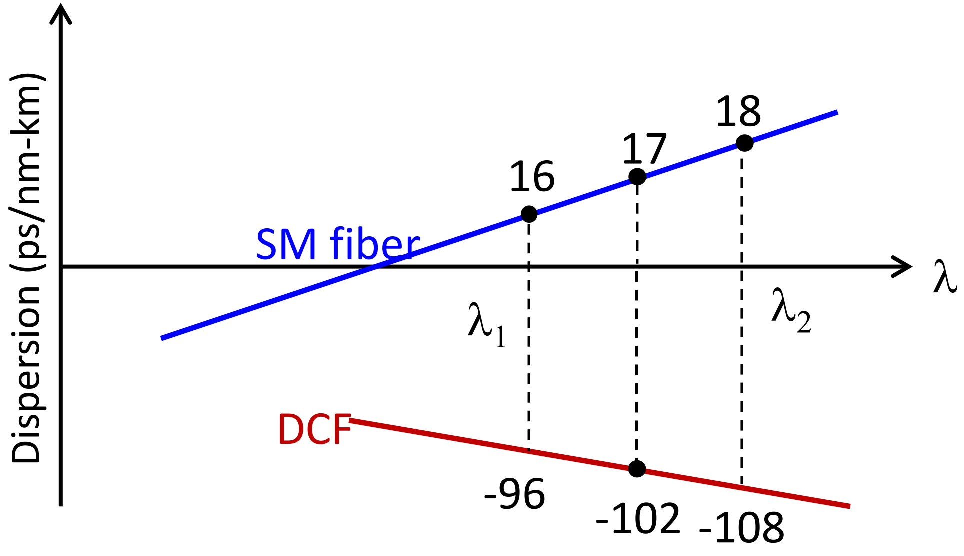
$$S_2 L_2 + S_1 L_1 = 0$$

To satisfy both conditions simultaneously:

$$\frac{S_2}{D_2} = \frac{S_1}{D_1}$$

The relative dispersion (S/D) slope for DCF and transmission fiber should be equal

Dispersion slope compensation

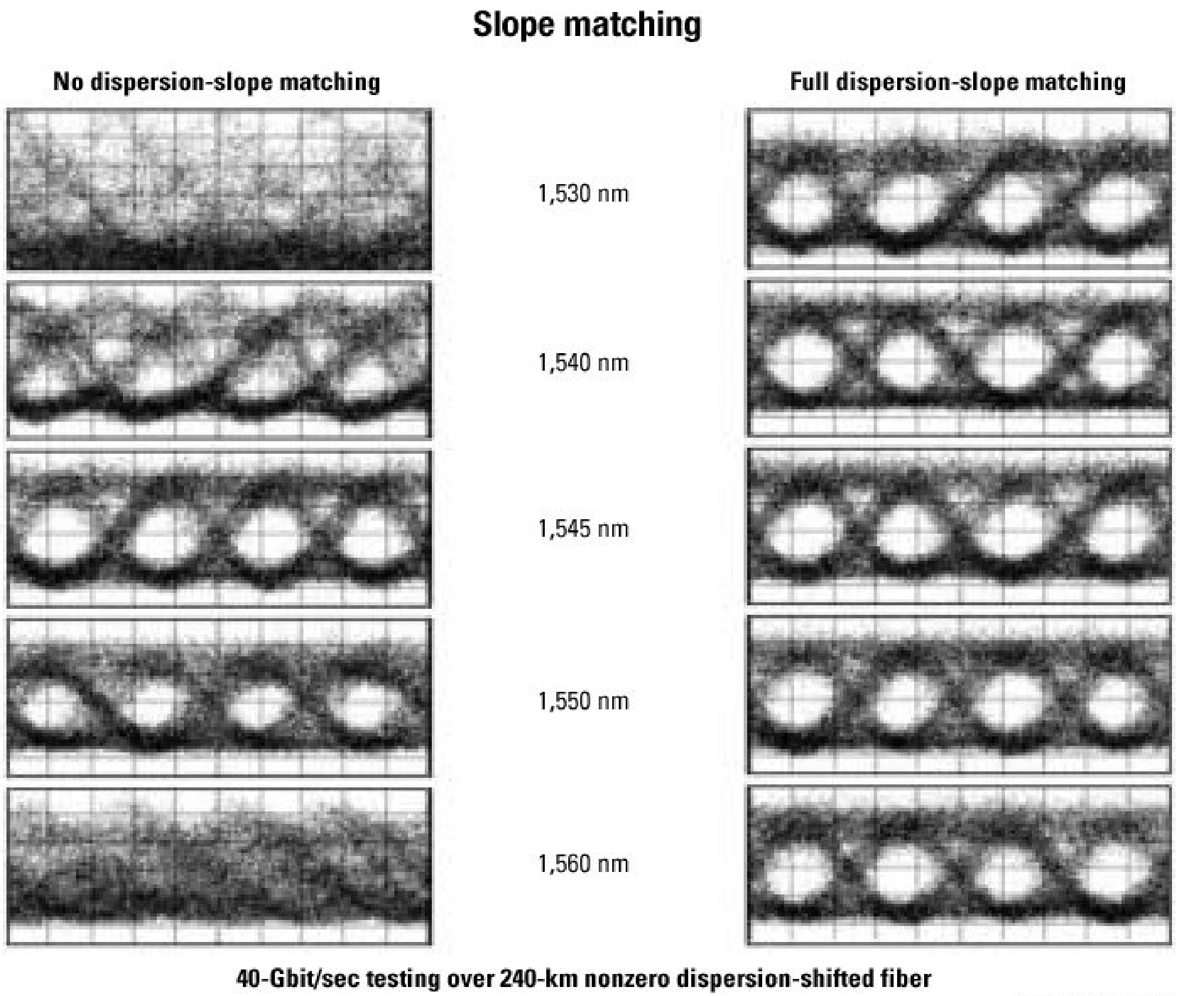


Within spectral window of interest ($\lambda_1 - \lambda_2$):

$$\frac{S_{DCF}}{D_{DCF}} = \frac{-12}{(\lambda_2 - \lambda_1)(-102)} \qquad \frac{S_{SMF}}{D_{SMF}} = \frac{2}{(\lambda_2 - \lambda_1)(17)} \Rightarrow \frac{S_{DCF}}{D_{DCF}} = \frac{S_{SMF}}{D_{SMF}}$$

Perfect compensation

Importance of slope matching



Source: AT&T Labs

Figure 2. Full slope match enables a clear optical signal across all wavelengths in the operating band. (Courtesy of AT&T Labs)

Example

You are given the following 3 fibers:

- SMF-28: dispersion of 17 ps/nm.km and slope of 0.057 ps/(nm^2.km) at 1550 nm
- Type A DCF: dispersion of -98 ps/nm.km and slope of -0.1 ps/(nm^2.km) at 1550 nm
- Type B DCF: dispersion of -134 ps/nm.km and slope of -0.5 ps/(nm^2.km) at 1550 nm

How much length of each type of DCF would you need to compensate the dispersion at 1550 nm from an 80 km link of SMF-28 ?

Using the length calculated in previous question, which fiber would do a better job at compensating multiple WDM channels centered at 1550 nm

Disadvantages of DCF

Added loss associated with the increased fiber span

Splices loss between fibers of different geometry

Nonlinear effects may degrade the signal over the long length of the fiber if the signal has sufficient intensity

Links that use DCF often require an additional amplifier stage to compensate for the added loss leading to increase noise

Other schemes have been developed: *dispersion equalizing filters*

Conclusions on budget and performance of transmission link

Attenuation and *dispersion* will each impose a *limitation* on the maximum length L of the link associated with a bit rate B

Limitations due to loss

- The power budget needs to be within the required operation of the system
- Enough optical power at the output of the link for a given performance, given all possible losses from the transmitter to the receiver (coupling loss, fiber loss etc ..)

Limitation due to dispersion

- Pulse broadens to a RMS value σ from initial σ_0 due to dispersion
- Interference will occur between pulses when σ becomes comparable to the time interval between bits $T_B = 1/B$
- Common criterion in terms of RMS used for maximum acceptable broadening:

$$\boxed{\sigma \leq \frac{1}{4B}}$$

σ, the root mean square width